Math 656 • FINAL EXAM • May 8, 2014

1) **(18pts)** Find and categorize **all zeros and singularities** of the following functions (you don't have to examine possible singularity at z=∞); make sure to explain briefly:

(a) $f(z)=\dfrac{\sin(1/z)}{(1+e^z)^2}$ (b) $f(z)=z^2\tanh\dfrac{1}{z}$ (c) $f(z)=\dfrac{z^{1/3}-1}{e^z-e}$

In (c), explain carefully the singularity at z=1; assume that the branch of $z^{1/3}$ satisfies $-\pi \le \arg z < \pi$

2) **(18pts)** Find the series representation of the following functions in the indicated regions:

(a) $f(z)=\dfrac{z\,e^z}{\sin^2 z}$ in $0<|z|<\pi$ (find the **first 3 dominant terms only**)

(b) $f(z)=\dfrac{z}{z^2-1}$ in $1<|z-2|<3$ (use partial fraction decomposition and the geometric series)

3) **(24pts)** Calculate the following integrals. Carefully explain each step, and make sure to obtain a **real** answer:

(a) $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2}$ (b) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos x\;dx}{x^2-2x+2}$ (c) $\displaystyle\int_{0}^{\infty}\frac{x^p\;dx}{x^4+1}$

In (c), find also the convergence condition on real constant p

4) **(12pts)** Use the Argument Principle to find the number of zeros of function $f(z)=z^7+i+2$ located within the sector $0\le \arg z<\dfrac{\pi}{4}$

5) **(12pts)** Show that the transformation $w=\dfrac{z-\alpha}{1-\bar{\alpha}z}$ (where α is a complex constant satisfying $|\alpha|<1$) maps a unit disk into itself (hint: examine the mapping of the **unit circle** by calculating $|w|^2$).

---------------------------------- **Choose between problems 6 and 7** ---

6) **(16pts)** Find the coefficients C_{-1}, C_{-2} and C_{-3} in the **principle part** of the Laurent series for $f(z)=\dfrac{z}{\sin z}$ converging within $\pi<|z|<2\pi$. This will help you in finding **all** coefficients in the principle part. Make sure to sketch the contour of integration and indicate all singularities before performing integration needed in finding the coefficients.

7) **(16pts)** Calculate $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ by integrating $f(z)=\dfrac{\pi}{z^2\sin(\pi z)}$ over a rectangular contour with sides formed by lines $\pm\left(N+\dfrac{1}{2}\right)+i\,y$ and $x\pm iN$ (where N is an integer) and then taking the limit $N\to\infty$